EXTREMAL EIGENVALUE PROBLEMS FOR CONVEX SETS OF SYMMETRIC MATRICES AND OPERATORS†

BY

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ABSTRACT

Let $A_1,\ldots,A_n$ and K be $m \times m$ symmetric matrices with K positive definite. Denote by C the convex hull of $\{A_1,\ldots,A_n\}$. Let $\{\lambda_p(KA)\}_1^n$ be the n real eigenvalues of KA arranged in decreasing order. We show that max $\lambda_p(KA)$ on C is attained for some $A^* = \Sigma_{i=1}^n \alpha_i^* A_i$ for which at most $p(p+1)/2$ of α_i^* do not vanish. We extend this result in several directions and consider applications to classes of integral equations.

1. Introduction

Let A and K be $m \times m$ real valued symmetric matrices. Assume furthermore that K is positive definite. Consider the matrix KA which is similar to the symmetric matrix $K^{\frac{1}{2}}AK^{\frac{1}{2}}$. Thus the eigenvalues of KA are real and we arrange them in decreasing order

$$\lambda_1(KA) \geqq \lambda_2(KA) \geqq \cdots \geqq \lambda_m(KA).$$

Denote by T the inverse matrix of K. It is a familiar fact that $\lambda_1(KA)$ can be characterized as

$$\lambda_1(KA) = \max_x \frac{x'Ax}{x'Tx}$$

where x is a column vector and x' the transposed line vector of x. Let $A_1,\cdots,A_n$ be $m \times m$ real valued symmetric matrices and denote by $C(A_1,\cdots,A_n)$ the con-

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vex hull of the set $\{A_1,\cdots,A_n\}$. Noting that $\lambda_1(KA)$ is a convex function on $C(A_1,\cdots,A_n)$ we see that the maximum of $\lambda_1(KA)$ is achieved on the set $\{A_1,\cdots,A_n\}$. This principle was used in [1] to derive a comparison theorem of a rather unusual type for second order linear differential equations. A natural question is to characterize a subset of $C(A_1,\cdots,A_n)$ on which $\lambda_p(KA)$ attains its maximum. In some cases this was done by Nowosad [8]. In order to state his results we recall a few definitions. A real valued $m\times m$ matrix R (not necessarly symmetric) is called totally positive (TP) if all its minors are nonnegative. If all minors of R are positive then R is called strictly totally positive (STP). The matrix R is said to be oscillating if R is TP and R^k is STP for some natural k. The remarkable properties of oscillating matrices are discussed in [3] and [4]. The eigenvalues of oscillating matrices are positive and distinct. Nowosad [8], relying on the fine structure of oscillating matrices established the following result:

THEOREM I. *Let K be an oscillating (not necessarily symmetric) matrix. Denote by $\mathscr{D}$ the set of all nonnegative diagonal matrices having trace of magnitude* 1. *Then* $\max_{D\in\mathscr{D}}\lambda_p(KD)$ *is achieved for a nonnegative diagonal matrix D with exactly p positive entries.*

This result is standardly extended to integral equations of the form

$$\int_0^1 K(x,y)\psi(y)d\rho(y) = \lambda\psi(x) \tag{1.1}$$

where ρ is a nonnegative measure normalized by the condition $\int_0^1 d\rho = 1$ and $K(x,y)$ is a continuous oscillating kernel. Recall that a continuous kernel $K(x,y)$ is said to be oscillating if for any $2m$ points $0 < x_1 < \cdots < x_m < 1$, $0 < y_1 < \cdots < y_m < 1$ the matrix $(K(x_i,y_j))_1^m$ is an oscillating matrix for $m = 1,2,\cdots$. It is known [3, p. 208] that for a nonnegative finite measure ρ the nontrivial eigenvalues of (1.1) are distinct and positive

$$\lambda_1(\rho) > \lambda_2(\rho) > \cdots > 0.$$

Nowosad proved

THEOREM II. *Let $K(x,y)$ be an oscillating (not necessarily symmetric) kernel. Let ρ be a nonnegative normalized measure $\int_0^1 d\rho = 1$. Then* $\max_\rho\lambda_p(\rho)$ *is achieved for a distribution $\bar\rho$ with exactly p concentrated masses.*

These results of Nowosad were extended in several directions by Karlin [6]. On the other hand, the case where $K(x,y)$ is the special oscillating positive definite kernel

$$K(x,y) = \begin{cases} x(1-y) & 0 \leqq x \leqq y \leqq 1, \\ (1-x)y & 0 \leqq y \leqq x \leqq 1 \end{cases} \tag{1.2}$$

was dealt with by a number of authors in different contexts. This kernel is the Green's function of the differential operator $-d^2u/dx^2$ associated with boundary conditions $u(0) = u(1)=0$. (Consult, for example, Karlin [6] and references therein.) Returning to our maximum problem

$$\max_{A \in C(A_1,\dots,A_n)} \lambda_p(KA)$$

and examining the examples above it is quite natural to consider the problem of the existence of an extremal matrix $\bar{A}$ which is a convex combination of s matrices $\{A_1,\cdots,A_n\}$ such that s depends only on p and not on n. To be more precise, let $H_j(A_1,\cdots,A_n)$ be the subset of $C(A_1,\cdots,A_n)$ of all matrices which are spanned by at most j matrices from $\{A_1,\cdots,A_n\}$. Under certain conditions we develop results of the form

$$\max_{A \in C(A_1,\dots,A_n)} \lambda_p(KA) = \max_{B \in H_s(A_1,\dots,A_n)} \lambda_p(KB)$$

such that the upper bound of s depends only on p. Our main result is

THEOREM 1. *Let $A_1,\cdots,A_n$ and K be $m \times m$ real valued symmetric matrices with K positive definite. Then*

$$\max_{A \in C(A_1,\dots,A_n)} \lambda_p(KA) = \max_{B \in H_{p(p+1)/2}(A_1,\dots,A_n)} \lambda_p(KB) \tag{1.3}$$

for $p = 1,\cdots,m$ and this result is best possible.

The set $C(A_1,\cdots,A_n)$ is called *totally nondegenerate with respect to the matrix* K if for any matrix belonging to this set, the inequalities

$$\lambda_1(KA) > \lambda_2(KA) > \cdots > \lambda_m(KA)$$

hold. Examples where this holds are indicated later. In this case we have:

THEOREM 2. *Let $A_1,\cdots,A_n$ and K be $m \times m$ real valued symmetric matrices with K positive definite. If the set $C(A_1,\cdots,A_n)$ is totally nondegenerate with respect to K, then*

$$\max_{A \in C(A_1,\dots,A_n)} \lambda_p(KA) = \max_{B \in H_p(A_1,\dots,A_n)} \lambda_p(KB) \tag{1.4}$$

for $p = 1,\cdots,n$.

The equality (1.4) holds for a specified p if any matrix belonging to the set

$C(A_1,\cdots,A_n)$ satisfies the condition $\lambda_{p-1}(KA) > \lambda_p(KA)$. Suppose that $J_1,\cdots,J_n$ are nonnegative symmetric positive definite Jacobi matrices. Then the set $C(J_1,\cdots,J_n)$ is totally nondegenerate with respect to a symmetric oscillating matrix K.

Theorems 1 and 2 can be extended in several ways. Some extensions are discussed in Section 4. For example, Nehari [7] considered the maximum problem

$$\max \frac{\lambda_p(A)}{(\int_0^1 A^{\frac{1}{2}}(x)dx)^2}$$

for the set of nonnegative nondecreasing functions $A(x)$, where $\lambda_p(A)$ are the eigenvalues of Sturm-Liouville system

$$\frac{d^2u}{dx^2} + \frac{1}{\lambda}A(x)u = 0, \quad u(0) = u(1) = 0.$$

In Section 4 we consider maximum of $\lambda_p(KA)/f(A)$ on the set $C(A_1,\cdots,A_n)$ where $f(A)$ is a concave positive function on this set. All of our results remain valid if $A_1,\cdots,A_n$ are bounded linear symmetric operators and K is a linear compact positive definite operator in a Hilbert space $\mathscr{H}$. The last section is mainly devoted to applications of our results to certain classes of integral equations.

2. The main result

Let $\mathscr{H}$ be an m-dimensional real Hilbert space with an inner product (x,y). Let A be a linear symmetric operator in $\mathscr{H}$, i.e. $(Ax,y) = (x,Ay)$. It is well known that all eigenvalues of A are real. We arrange them in decreasing order

$$\lambda_1(A) \geqq \lambda_2(A) \geqq \cdots \geqq \lambda_m(A). \tag{2.1}$$

Furthermore to each eigenvalue λ_i corresponds an eigenvector x_i satisfying

$$Ax_i = \lambda_i x_i \qquad i = 1,\cdots,m,$$

$$(x_i,x_j) = \delta_{ij}, \quad i,j = 1,\cdots,m.$$

There are several known extremal characterization of the eigenvalues $\lambda_i\ (A)$ (see for example [2, I p. 321] and [9]). For our later applications we need the following max-min characterization going back to Poincaré and which Pólya and Schiffer called the Convoy Principle [9]:

LEMMA 1. *Let A be a linear symmetric operator in an m-dimensional* *real Hilbert space. Denote by $\lambda_1(A) \geqq \cdots \geqq \lambda_m(A)$ the m real eigenvalues of A arranged in decreasing order. Then each λ_i is given as*

$$\lambda_i = \max_{S_i} \min_{x \in S_i} \frac{(Ax, x)}{(x, x)}, \tag{2.2}$$

where S_i is a vector subspace of a dimension i. The maximum is obtained for the subspace S_i spanned by the first i eigenvectors $x_1, \cdots, x_i$. However, the maximum may be obtained for other subspaces S_i. If the minimum of $(Ax,x)/(x,x)$ on some subspace S_i is equal to λ_i then S_i contains an eigenvector of A corresponding to λ_i.

Let A be an $m \times m$ real valued symmetric matrix. The matrix A is a linear symmetric operator on a space of all m-dimensional real valued column vectors with respect to the inner product

$$(x, y) = y'x, \tag{2.3}$$

where y' is the transposed line vector of y. Now the eigenvalues of A can be characterized by the convoy principle. Let $A_1, \cdots, A_n$ be $m \times m$ real valued symmetric matrices. Denote by $C(A_1, \cdots, A_n)$ the smallest convex set which contains $A_1, \cdots, A_n$:

$$C(A_1, \cdots, A_n) = \{A \mid A = \sum_{i=1}^{n} \alpha_i A_i,\ \alpha_i \geqq 0, \quad i = 1, \cdots, n,\ \sum_{i=1}^{n} \alpha_i = 1\}.$$

By P_n we denote the set of all n-dimensional probability vectors:

$$P_n = \{\alpha = (\alpha_1, \cdots, \alpha_n) \mid \alpha_i \geqq 0,\ i = 1, \cdots, n,\ \sum_{i=1}^{n} \alpha_i = 1\}.$$

Assume that A belongs to $C(A_1, \cdots, A_n)$. We say that A is represented by $\alpha = (\alpha_1, \cdots, \alpha_n)$ if $A = \sum_{i=1}^{n} \alpha_i A_i$, and $\alpha \in P_n$. Clearly the representation of A may not be unique. By $P_{n,j}$ we denote the set of all n-dimensional probability vectors which have at most j nonvanishing components:

$$P_{n,j} = \{\alpha = (\alpha_1, \cdots, \alpha_n) \mid \alpha \in P_n,\ \sum_{k=1}^{j} \alpha_{i_k} = 1,\ \text{for } 1 \leqq i_1 < i_2 \cdots < i_j \leqq n\}.$$

Let $H_j(A_1, \cdots, A_n)$ be the set of all matrices spanned by at most j matrices from $\{A_1, \cdots, A_n\}$:

$$H_j(A_1, \cdots, A_n) = \{A \mid A = \sum_{i=1}^{n} \alpha_i A_i,\ (\alpha_1, \cdots, \alpha_n) \in P_{n,j}\}.$$

For $j \geqq n$ $H_j(A_1, \cdots, A_n)$ is defined as the set $C(A_1, \cdots, A_n)$. Let K be an $m \times m$

real valued symmetric positive definite matrix. Consider the matrix KA where A belongs to the set $C(A_1, \cdots, A_n)$. The matrix KA is similar to the symmetric matrix $K^{\frac{1}{2}}AK^{\frac{1}{2}}$ since $K^{\frac{1}{2}}AK^{\frac{1}{2}} = K^{-\frac{1}{2}}(KA)K^{\frac{1}{2}}$. Thus the eigenvalues of KA are real and we arrange them in decreasing order $\lambda_1(KA) \geqq \cdots \geqq \lambda_m(KA)$. As we pointed out earlier:

$$\lambda_i(KA) = \lambda_i(K^{\frac{1}{2}}AK^{\frac{1}{2}}) \tag{2.4}$$

for $i = 1, \cdots, m$. Consider the following maximum problem defined on $C(A_1, \cdots, A_n)$: Find $\max_{A \in C(A_1, \ldots, A_n)} \lambda_p(KA)$ for $p = 1, \cdots, m$.

DEFINITION 1. A natural number d_p is called the pth degree of the set $C(A_1, \cdots, A_n)$ with respect to the maximal problem $\max_{A \in C(A_1, \ldots, A_n)} \lambda_p(KA)$ if

$$\max_{A \in C(A_1, \ldots, A_n)} \lambda_p(KA) = \max_{B \in H_{d_p}(A_1, \cdots A_n)} \lambda_p(KB)$$

and

$$\max_{A \in C(A_1 \ldots A_n)} \lambda_p(KA) > \max_{B \in H_{d_p - 1}(A_1 \ldots A_n)} \lambda_p(KB)$$

if $d_p > 1$.

By definition $1 \leqq d_p \leqq n$.

THEOREM 1. *Let $A_1, \cdots, A_n$ and K be $m \times m$ real valued symmetric matrices with K positive definite. Then*

$$\max_{A \in C(A_1, \ldots, A_n)} \lambda_p(KA) = \max_{B \in H_{p(p+1)/2}(A_1, \ldots, A_n)} \lambda_p(KB) \tag{2.5}$$

for $p = 1, \cdots, m$ and this result is best possible.

PROOF. Consider first the case where K is an identity matrix I. Assume to the contrary that the p-degree of the set $C(A_1, \cdots, A_n)$ is greater than $p(p+1)/2$, i.e., $d_p \geqq p(p+1)/2 + 1$. Let A^* be an extremal matrix

$$\max_{A \in C(A_1, \ldots, A_n)} \lambda_p(A) = \lambda_p(A^*) = \lambda_p^* \tag{2.6}$$

belonging to the set $H_{d_p}(A_1, \cdots, A_n)$. That means there exists a representation $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*)$ with nonvanishing components $\alpha_{i_j}^*$, $j = 1, \cdots, d_p$ and $1 \leqq i_1 < i_2 \cdots < i_{d_p} \leqq n$.

By the convoy principle

$$\lambda_p(A^*) = \min_{x \in S_p} \frac{(A^*x, x)}{(x, x)}$$

for some p-dimensional subspace, where the inner product (x, y) is defined by

(2.3). Furthermore S_p can be chosen as a subspace spanned by the p first eigenvectors $x_1, \cdots, x_p$ of A^*. Consider the following homogeneous system of linear equation in the d_p unknowns $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{d_p}$:

$$\sum_{j=1}^{d_p} \tilde{\alpha}_j (A_{i_j} x_u, x_v) = 0 \tag{2.7a}$$

for all $u, v \leqq p$ except for $u = v = p$, and

$$\sum_{j=1}^{d_p} \tilde{\alpha}_j = 0. \tag{2.7b}$$

These are exactly $p(p+1)/2$ equations with d_p unknowns. As $d_p > p(p+1)/2$ there exists a nontrivial solution $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{d_p}$.

Form $\tilde{A} = \sum_{j=1}^{d_p} \tilde{\alpha}_j A_{i_j}$ and $A_\varepsilon = A^* + \varepsilon \tilde{A}$ with ε to be determined. For any $y \in S_p$, i.e. $y = \sum_{i=1}^{p} a_i x_i$ and $(y, y) = 1$ we have in view of (2.7)

$$(A_\varepsilon y, y) = (A^* y, y) + \varepsilon a_p^2 (\tilde{A} x^{(p)}, x^{(p)}).$$

Choose the sign of ε so that

$$\varepsilon(\tilde{A} x^{(p)}, x^{(p)}) \geqq 0. \tag{2.8}$$

Then

$$(A_\varepsilon y, y) \geqq \lambda_p(A^*) = \lambda_p^*.$$

From the convoy principle we conclude that

$$\lambda_p(A_\varepsilon) \geqq \lambda_p^*. \tag{2.9}$$

Now for $|\varepsilon|$ small enough, the matrix A_ε belongs to the set $H_{d_p}(A_1, \cdots, A_n)$. Increase $|\varepsilon|$ in magnitude from 0 preserving the sign in (2.8) until the first $\alpha_{i_j}^* + \varepsilon \tilde{\alpha}_j$, $j = 1, \cdots, d_p$, vanishes. This holds for some positive $|\varepsilon_0|$ in view of (2.7b). Thus we obtained that A_{ε_0} is in $H_{d_p - 1}(A_1, \cdots, A_n)$. From (2.9) we have that $\lambda_p(A_{\varepsilon_0}) = \lambda_p^*$ and finally we conclude that the p-degree of the set $C(A_1, \cdots, A_n)$ is not greater than $d_p - 1$. This contradiction proves the equality (2.5) in case that $K = I$. For a general positive definite matrix K we use the equalities (2.4) to obtain the above case.

The following example shows that the upper bound $p(p+1)/2$ in (2.5) is best possible. First let $p = m$. For fixed integers α, β satisfying $1 \leqq \alpha \leqq m$, $1 \leqq \beta \leqq m$ define an $m \times m$ symmetric matrix $E_{\alpha\beta} = (e_{i\,j}^{\alpha\,\beta})_1^m$ to be

$$e_{i\,j}^{\alpha\,\beta} = \begin{cases} 1 & \text{for } i = \alpha,\ j = \beta, \\ 1 & \text{for } i = \beta,\ j = \alpha, \\ 0 & \text{for other values of } i \text{ and } j. \end{cases}$$

Let $Q_1, \cdots, Q_{m(m+1)/2}$ be $m \times m$ symmetric matrices $Q_1 = E_{11} + E_{12}$; $Q_2 = E_{11} - E_{12} + E_{13}, \cdots$; $Q_m = E_{11} - E_{1m} + E_{23}, \cdots$; $Q_{2m-3} = E_{11} - E_{2(m-1)} + E_{2m}, \cdots$; $Q_{m(m-1)/2-1} = E_{11} - E_{(m-2)(m-1)} + E_{(m-2)m}$; $Q_{m(m-1)/2} = E_{11} - E_{(m-2)m} + E_{(m-1)m}$; $Q_{m(m-1)/2+1} = E_{11} - E_{(m-1)m}$; $Q_{m(m-1)+i} = E_{ii}$, $i = 2, \cdots, m$.

The trace of each $E_{\alpha\beta}$ is equal to $\delta_{\alpha\beta}$ and so trace of each Q_i is equal to 1. This shows that $\mathrm{tr}(Q) = 1$ for any $Q \in C(Q_1, \cdots, Q_{m(m+1)/2})$. Let K be the unit matrix I. It is well known that

$$\sum_{i=1}^{m} \lambda_i(Q) = \mathrm{tr}(Q) = 1, \quad Q \in C(Q_1, \cdots, Q_{m(m+1)/2}).$$

Therefore $\lambda_m(Q) \leqq 1/m$ with the sign of equality holding only if

$$\lambda_1(Q) = \lambda_2(Q) = \cdots = \lambda_m(Q).$$

As Q is a symmetric matrix, the equality $\lambda_m(Q) = 1/m$ holds if and only if $Q = (1/m)I$. It is easy to show that

$$\frac{1}{m} = \sum_{i=1}^{m(m+1)/2} \alpha_i Q_i$$

only if

$$\alpha_i = \frac{1}{m[m(m-1)/2+1]}, \text{ for } i = 1, \cdots, m(m-1)/2 + 1,$$

$$\alpha_i = \frac{1}{m}, \quad \text{for } i = m(m-1)/2 + 2, \cdots, m(m+1)/2.$$

This shows that the m-degree of the set $C(Q_1, \cdots, Q_{m(m+1)/2})$ is exactly $m(m+1)/2$. For $1 \leqq p < m$ let $Q_1, \cdots, Q_{p(p+1)/2}$ be the $p \times p$ symmetric matrices defined above. We define an $m \times m$ matrix A_i as a diagonal block matrix $A_i = \mathrm{diag}\{Q_i, 0\}$, for $i = 1, \cdots, p(p+2)/2$. For any matrix A, $C(A_1, \cdots, A_{p(p+1)/2})$ is a block diagonal matrix $\mathrm{diag}\{Q, 0\}$ where Q belongs to $C(Q_1, \cdots, Q_{p(p+1)/2})$. According to the previous case

$$\max_{A \in C(A_1, \ldots, A_{p(p+1)/2})} \lambda_p(A) = \max_{Q \in C(Q_1, \ldots, Q_{p(p+1)/2})} \lambda_p(Q) = \frac{1}{p}$$

and this maximum is obtained for a unit matrix of the form $\mathrm{diag}\{(1/p)I_p, 0\}$ belonging only to the set $H_{p(p+1)/2}(A_1, \cdots, A_{p(p+1)/2})$ (I_p is a $p \times p$ unit matrix). The proof of Theorem 1 is completed.

In what follows we show that if for a some p Theorem 1 is sharp then there exist an extremal matrix B which satisfies

$$\lambda_1(KB) = \lambda_2(KB) = \cdots = \lambda_p(KB).$$

Thus if no matrix A in $C(A_1, \cdots, A_n)$ satisfies these conditions we can improve the upper bound $p(p+1)/2$.

3. Totally nondegenerate sets

The set $C(A_1, \cdots, A_n)$ is called totally nondegenerate with respect to the matrix K if any matrix A from this set satisfies the inequalities

$$\lambda_1(KA) > \lambda_2(KA) > \cdots > \lambda_m(KA). \tag{3.1}$$

THEOREM 2. *Let $A_1, \cdots, A_n$ and K be $m \times m$ real valued symmetric matrices with K positive definite. If the set $C(A_1, \cdots, A_n)$ is totally nondegenerate with respect to K, then*

$$\max_{A \in C(A_1,\ldots,A_n)} \lambda_p(KA) = \max_{B \in H_p(A_1,\ldots,A_n)} \lambda_p(KB) \tag{3.2}$$

for $p = 1, \cdots, m$.

PROOF. As in the proof of Theorem 1 assume that $K = I$. Let $\mathscr{E}_p$ be the set of all extremal matrices B, i.e.,

$$\max_{A \in C(A_1,\ldots,A_n)} \lambda_p(A) = \lambda_p(B).$$

The set $\mathscr{E}_p$ is compact. Let P_n be the set of all probability vectors $\alpha = (\alpha_1, \cdots, \alpha_n)$. Denote by $\alpha^+ = (\alpha_1^+, \cdots, \alpha_n^+)$ an increasing rearrangement of α: $\alpha_1^+ \leqq \alpha_2^+ \leqq \cdots \leqq \alpha_n^+$, $\alpha_j^+ = \alpha_{i_j}$, $j = 1, \cdots, n$ and $\{i_1, \cdots, i_n\}$ is the set $\{1, \cdots, n\}$. Define a complete ordering in P_n: $\alpha \prec \beta$ iff

$$\alpha_j^+ = \beta_j^+, \ j = 1, \cdots, i-1, \text{ and } \alpha_i^+ < \beta_i^+ \text{ where } 1 \leqq i \leqq n. \tag{3.3}$$

The set $\bar{\mathscr{E}}_p$ is the collection of all representations in P_n of the extremal matrices belonging to $\mathscr{E}_p$:

$$\bar{\mathscr{E}}_p = \{\alpha = (\alpha_1, \cdots, \alpha_n) \mid \alpha \in P_n, \ \sum_{i=1}^n \alpha_i A_i \in \mathscr{E}_p\}. \tag{3.4}$$

Again $\bar{\mathscr{E}}_p$ is a compact set. Let $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*)$ be a minimal element in $\bar{\mathscr{E}}_p$ with respect to a complete order (3.3). Define A^* to be the matrix $\sum_{i=1}^n \alpha_i^* A_i$. Clearly A^* belongs to $\mathscr{E}_p$. We claim that A^* is in $H_p(A_1, \cdots, A_n)$. Assume to the contrary that $0 < \alpha_{i_j}^*$ for $j = 1, \cdots, r$ with $r \geqq p+1$ and other α_i^* vanish. Consider the nontrivial solution $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_r$ of p linear homogenic equations

$$(3.5)\qquad \sum_{j=1}^{r} \tilde{\alpha}_j(A_{i_j}x_u, x_p) = 0, \qquad u = 1,\cdots,p-1$$

$$\sum_{j=1}^{r} \tilde{\alpha}_j = 0.$$

The vectors $x_1,\cdots,x_p$ are the p first orthonormal eigenvectors of A^*. Form $\tilde{A} = \sum_{j=1}^{r}\tilde{\alpha}_j A_{i_j}$ and $A_\varepsilon = A^* + \varepsilon\tilde{A}$. Let $y \in S_p = \mathrm{sp}\{x_1,\cdots,x_p\}$. Then y is of the form $y = z + ax_p$ where $z \in \mathrm{sp}\{x_1,\cdots,x_{p-1}\}$. From (3.5) it follows that $(Ax_u, x_p) = 0$ for $u = 1,\cdots,p-1$. Thus

$$(A_\varepsilon y, y) = (A^*z, z) + a^2\lambda_p^* + \varepsilon(\tilde{A}z, z) + \varepsilon a^2(\tilde{A}x_p, x_p).$$

Now $(|\tilde{A}z, z)| \leqq b(z,z)$ for some positive b. Noting that $(A^*z, z) \geqq \lambda_{p-1}(z,z)$ we obtain that

$$(A^*z, z) + a^2\lambda_p^* + \varepsilon(\tilde{A}z, z) + \varepsilon a^2(\tilde{A}x_p, x_p) \geqq (\lambda_{p-1} - |\varepsilon|\, b)(z,z) + a^2\lambda_p^* + \varepsilon a^2(\tilde{A}x_p, x_p).$$

The nondegeneracy of the set $C(A_1,\cdots,A_n)$ implies that $\lambda_{p-1} > \lambda_p^*$. For a small enough ε of the sign $(\tilde{A}x_p, x_p)$ such that:

$$\lambda_{p-1} - |\varepsilon|\, b > \lambda_p^*, \quad A_\varepsilon \in C(A_1,\cdots,A_n)$$

we have the inequality

$$\lambda_p(A_\varepsilon) \geqq \min_{y\in S_p} \frac{(A_\varepsilon y, y)}{(y,y)} \geqq \lambda_p^*.$$

But $\lambda_p^* \geqq \lambda_p(A_\varepsilon)$, therefore it follows that

$$\lambda_p(A_\varepsilon) = \min_{y\in S_p} \frac{(A_\varepsilon y, y)}{(y,y)} = \lambda_p^*$$

and this equality requires that $(\tilde{A}x_p, x_p) = 0$. So $\lambda_p(A_\varepsilon) = \lambda_p^*$ for a small enough ε of arbitrary sign. Let $\alpha(\varepsilon) = (\alpha_1(\varepsilon),\cdots,\alpha_n(\varepsilon))$ be the vector $\alpha_{i_j}(\varepsilon) = \alpha_{i_j}^* + \varepsilon\tilde{\alpha}_{i_j}$, $j = 1,\cdots,r$ and $\alpha_i(\varepsilon) = 0$ for other components of $\alpha(\varepsilon)$. As we showed above, $\alpha(\varepsilon)$ belongs to the set $\bar{\mathscr{E}}_p$ for any small ε of an arbitrary sign. This contradicts the minimality of α^* in $\bar{\mathscr{E}}_p$ with respect to a complete order (3.3). The proof of the theorem is completed.

Examining the proof of Theorem 2 we see that (3.2) holds for some p if any A belonging to $C(A_1,\cdots,A_n)$ satisfies

$$(3.6)\qquad \lambda_{p-1}(KA) > \lambda_p(KA).$$

The set $C(A_1,\cdots,A_n)$ is called nondegenerate of order p with respect to K if (3.6) holds for each A from this set. The set $C(A_1,\cdots,A_n)$ is called approximable by nondegenerate set of order p with respect to K if

i) there exists a sequence of $n+1$ symmetric $m \times m$ matrices $A_{1q},\cdots,A_{nq}$, K_q, $q = 1,2,\cdots$, such that $\lim_{q\to\infty} A_{iq} = A_i$, $i = 1,\cdots,n$, $\lim_{q\to\infty} K_q = K$,

ii) the sets $C(A_{1q},\cdots,A_{nq})$ are nondegenerate of order p with respect to K_q. If the sets $C(A_{1q},\cdots,A_{nq})$ are totally nondegenerate with respect to K_q then $C(A_1,\cdots,A_n)$ is called approximable by a totally nondegenerate set with respect to K. In fact we proved:

THEOREM 3. *Let $A_1,\cdots,A_n$ and K be $m \times m$ real valued symmetric matrices with K positive definite. Let the set $C(A_1,\cdots,A_n)$ be a nondegenerate of order p with respect to K or approximable by such a set. Then*

$$\max_{A\in C(A_1,\dots,A_n)} \lambda_p(KA) = \max_{B\in H_p(A_1,\dots,A_n)} \lambda_p(KB). \tag{3.7}$$

If the set $C(A_1,\cdots,A_n)$ is totally nondegenerate with respect to K or approximable by such a set, then this equality holds for $p = 1,\cdots,m$.

We now bring an example of a totally nondegenerate set. Recall that a matrix Q is called totally positive (TP) if all minors of Q are nonnegative. If all minors of Q are positive then Q is called strictly totally positive (STP). If Q is a TP matrix and for some natural power i Q is a STP matrix, then Q is called an oscillating matrix. It is known [3, p. 123] that the product of an oscillating matrix with a nonsingular totally positive matrix is again an oscillating matrix. The eigenvalues of an oscillating matrix are positive and distinct.

For further remarkable properties of these matrices see [2], [3] and [4]. Let $J = (h_{ij})_1^m$ be a Jacobi matrix, i.e., $h_{ij} = 0$ for $|i-j| \geqq 2$. If J is a nonnegative symmetric matrix which is nonnegative definite, then J is a totally positive matrix [4, p. 113]. Suppose that $J_1,\cdots,J_n$ are n nonnegative positive definite Jacobi $m \times m$ matrices, and let K be a symmetric oscillating matrix. It follows that the set $C(J_1,\cdots,J_n)$ is totally nondegenerate with respect to the matrix K. In [3, p. 316] it is shown that any nonsingular TP matrix can be obtained as a limit of STP matrices. Therefore, if $J_1,\cdots,J_n$ are nonnegative Jacobi matrices which are nonnegative definite, then $C(J_1,\cdots,J_n)$ is approximable by a totally nondegenerate set with respect to a positive definite TP matrix K.

COROLLARY 1. *Let $J_1,\cdots,J_n$ be $m \times m$ nonnegative Jacobi matrices which* *are symmetric and nonnegative definite. Then, for a positive definite TP matrix* K, *we have the equalities*

$$(3.8) \qquad \max_{J \in C(J_1,\dots,J_n)} \lambda_p(KJ) = \max_{J \in H_p(J_1,\dots,J_n)} \lambda_p(K\bar{J})$$

for $p = 1,\cdots,m$.

Let K be a positive definite matrix such that all its minors of order $p-1$ are positive. If J is a nonnegative and positive definite Jacobi matrix, then it is easy to see that $p-1$ compound of the matrix KJ is positive. Since the spectrum of KJ is positive from the Perron-Frobenius theorem, it follows that

$$\lambda_{p-1}(KJ) > \lambda_p(KJ).$$

This inequality implies that the set $C(J_1,\cdots,J_n)$ is nondegenerate of order p with respect to K, where $J_1,\cdots,J_n$ are nonnegative and positive definite Jacobi matrices. Using the continuity principle we have by Theorem 3:

COROLLARY 2. *Let* $J_1,\cdots,J_n$ *be nonnegative Jacobi matrices which are nonnegative definite. Let* K *be a positive definite matrix with minors of order* $p-1$ *all nonnegative. Then*

$$3.9) \qquad \max_{J \in C(J_1,\dots,J_n)} \lambda_p(KJ) = \max_{J \in H_p(J_1,\dots,J_n)} \lambda_p(K\bar{J}).$$

4. Extensions

Let $A_1,\cdots,A_n$ and K be $n+1$ symmetric matrices with K being positive definite.

DEFINITION 2. A natural number μ_p is called the degeneracy index of order p of the set $C(A_1,\cdots,A_n)$ with respect to K if any matrix from this set satisfies the inequality

$$\lambda_{p-\mu_p}(KA) > \lambda_p(KA).$$

Furthermore, there exists a matrix A belonging to $C(A_1,\cdots,A_n)$ for which the equality

$$\lambda_{p-\mu_p+1}(KA) = \lambda_p(KA)$$

holds. For $p = 1$, μ_1 is defined to be 1.

DEFINITION 3. A natural number ν_p is called the extremal degeneracy index of order p of the set $C(A_1,\cdots,A_n)$ with respect to K if any extremal matrix A^*, i.e., $\max_{A \in C(A_1,\dots,A_n)} \lambda_p(KA) = \lambda_p(KA^*)$, satisfies

$$\lambda_{p-\nu_-}(KA^*) > \lambda_p(KA^*).$$

Furthermore, there exists an extremal matrix A^* for which the equality

$$\lambda_{p-\nu p+1}(KA^*) = \lambda_p(KA^*)$$

holds. For $p = 1$, ν_1 is defined to be 1.

By definition we have

$$1 = \nu_p \leqq \mu_p \leqq p. \tag{4.1}$$

THEOREM 4. *Let $A_1, \cdots, A_n$ and K be $m \times m$ real valued symmetric matrices with K positive definite. Denote by μ_p the degeneracy index of order p of the set $C(A_1, \cdots, A_n)$ with respect to K. Then*

$$\max_{A \in C(A_1, \dots, A_n)} \lambda_p(KA) = \max_{B \in H\mu_p(2p-\mu_p+1)/2(A_1, \dots, A_n)} \lambda_p(KB) \tag{4.2}$$

for $p = 1, \cdots, m$. Moreover, if the extremal degeneracy index ν_p is strictly less than μ_p for some p then

$$\max_{A \in C(A_1, \cdots, A_n)} \lambda_p(KA) = \max_{B \in H\nu_p(2p-\nu_p+1)/2(A_1, \dots, A_n)} \lambda_p(KB). \tag{4.3}$$

PROOF. The proof of this theorem is merely a repetition of the proof of Theorem 2 with the following modifications: Let A^* be the extremal matrix defined in the proof of Theorem 2. Assume that ν is the degeneracy index of order p of A^*.

$$\lambda_{p-\nu}(A^*) > \lambda_{p-\nu+1}(A^*) = \cdots = \lambda_p(A^*). \tag{4.4}$$

(Note if $\nu = p$ then $\nu = \nu_p = p$ and (4.3) reduces to (2.5). Thus we assume that $\nu > p$). We claim that A^* belongs to $H_{\nu(2p-\nu+1)/2}(A_1, \cdots, A_n)$. As $\nu \leqq \nu_p$ this implies that $A^* \in H_{\nu_p(2p-\nu_p+1)/2}(A_1, \cdots, A_n)$ which proves (4.3). Assume to the contrary that $0 < \alpha^*_{i_j}$ for $j = 1, \cdots, r$ with $r \geqq \nu(2p-\nu+1)/2 + 1$ and other α^*_i vanish. Consider the nontrivial solution $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_r$ of $\nu(2p-\nu+1)/2$ linear homogenic equations

$$\sum_{j=1}^{r} \tilde{\alpha}_j(A_{i_j}x_u, x_p) = 0 \qquad u = 1, 2, \cdots, p-1,$$

$$\sum_{j=1}^{r} \tilde{\alpha}_j(A_{i_j}x_u, x_v) = 0, \qquad v = p-1, \cdots, p-\nu+1,\ u = 1, \cdots, v, \tag{4.5}$$

$$\sum_{j=1}^{r} \tilde{\alpha}_j = 0.$$

Form $\tilde{A} = \sum_{j=1}^{r} \tilde{\alpha}_j A_{i_j}$ and $A_\varepsilon = A^* + \varepsilon \tilde{A}$.

As in the proof of Theorem 2 we obtain that $\lambda_p(A_\varepsilon) = \lambda_p^*$ for a small enough ε of an arbitrary sign. This contradicts the minimality of α^*.

We remark that if Theorem 1 is sharp for some p then by Theorem 4 we have the equality $v_p = p$. This implies the existence of an extremal matrix B such that $\lambda_1(KB) = \cdots = \lambda_p(KB)$ as we claimed at the end of Section 2.

It is worth emphasizing that we have no examples of sets whose degeneracy index μ_p and extremal index v_p satisfy the inequalities

$$1 < v_p \leqq \mu_p < p.$$

From here we shall state our theorems only for the degeneracy index μ_p and all our results will remain true if we replace μ_p by v_p.

Consider next the maximum problem: Find

$$\max_{A \in C(A_1, \dots, A_n)} \lambda_p(KA)/f(A) \tag{4.6}$$

for a positive concave function f (see Section 1 for motivation of this problem). We define exactly what we mean by a continuous function on $C(A_1, \cdots, A_n)$.

Definition 4. A function $g(A)$ is a continuous function on the set $C(A_1, \cdots, A_n)$ if there exists an appropriate continuous function $g(\alpha)$ defined on P_n such that $g(A) = g(\alpha)$ where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a representation of A, i.e., $A = \sum_{i=1}^n \alpha_i A_i$. The function $g(A)$ is linear if $g(\alpha)$ is linear and $g(A)$ is concave on $C(A_1, \cdots, A_n)$ if $g(\alpha)$ is concave on P_n.

Note that by this definition we may have that $g(A)$ is multi alent in the case that A has more than one representation $A = \sum_{i=1}^n \alpha_i A_i = \sum_{i=1}^n \beta_i A_i$ and $g(\alpha) \neq g(\beta)$.

Theorem 5. *Let $A_1, \cdots, A_n$ and K be $m \times m$ real valued symmetric nonnegative definite matrices with K positive definite. Denote by μ_p the degeneracy index of order p of the set $C(A_1, \cdots, A_n)$. Let f be a positive concave function defined on $C(A_1, \cdots, A_n)$. Then*

$$\max_{A \in C(A_1, \dots, A_n)} \lambda_p(KA)/f(A) = \max_{B \in H\mu_p(2p-\mu_p+1)/2(A_1, \dots, A_n)} \lambda_p(KB)f(B) \tag{4.7}$$

for $p = 1, \cdots, m$.

To prove the theorem consider an auxiliary lemma:

Lemma 2. *Let $A_1, \cdots, A_n$ and K be $m \times m$ real valued symmetric matrices with K positive definite. Let f be a positive linear function on $C(A_1, \cdots, A_n)$. Then*

$$\max_{A \in C(A_1, \cdots, A_n)} \lambda_p(KA)/f(A) = \max_{B \in H_{\mu_n(2_n - \mu_n + 1)}(A_1, \ldots, A_n)} \lambda_p(KB)/f(B), p = 1, \cdots, m.$$

PROOF. The positive linear function f is of the form

$$f\left(\sum_{i=1}^{n} \alpha_i A_i\right) = \sum_{i=1}^{n} \theta_i \alpha_i$$

where $\theta_i > 0$ or $i = 1, \cdots, n$. Define a matrix $\bar{A}_i$ to be A_i/θ_i for $i = 1, \cdots, n$. Since the function $\lambda_p(KA)$ is homogenic it is easy to show that for any matrix A belonging to the set $C(A_1, \cdots, A_n)$ the equality $\lambda_p(KA)/f(A) = \lambda_p(K\bar{A})$ holds for an appropriate matrix $\bar{A}$ from $C(\bar{A}_1, \cdots, \bar{A}_n)$ and vice versa. In fact, if A and $\bar{A}$ have representations $A = \sum_{i=1}^{n} \alpha_i A_i$ and $\bar{A} = \sum_{i=1}^{n} \bar{\alpha}_i \bar{A}_i$ then $\bar{\alpha}_1, \cdots, \bar{\alpha}_n$ are defiined by the equations

$$\bar{\alpha}_i = \alpha_i \theta_i \Big/ \sum_{j=1}^{n} \alpha_j \theta_j, \qquad i = 1, \cdots, n. \tag{4.8}$$

The transformation above is a one-to-one transformation of P_n onto itself. Furthermore, if α has exactly k vanishing components then $\bar{\alpha}$ has the same number of vanishing components. Thus

$$\max_{A \in C(A_1, \ldots, A_n)} \lambda_p(KA)/f(A) = \max_{\bar{A} \in C(\bar{A}_1, \ldots, \bar{A}_n)} \lambda_p(K\bar{A}).$$

Applying Theorem 4 to the maximum problem on the right hand side and carrying out the inverse transformation of (4.8) we establish (4.7) for a linear positive f.

PROOF OF THEOREM 5. Let f be a positive concave function on $C(A_1, \cdots, A_n)$. Let $\mathscr{L}$ denote all linear functions on $C(A_1, \cdots, A_n)$ such that $L(B) \geqq f(B)$ for any $B \in C(A_1, \cdots, A_n)$. Note that L is positive on $C(A_1, \cdots, A_n)$ since f is positive. It is well known that $f(A)$ may be characterized as

$$f(A) = \min_{L \in \mathscr{L}} L(A).$$

The assumption that $A_1, \cdots, A_n$ are nonnegative definite implies that $\lambda_p(KA) \geqq 0$ and thus

$$\frac{\lambda_p(KA)}{f(A)} = \max_{L \in \mathscr{L}} \frac{\lambda_p(KA)}{L(A)}.$$

Now

$$\max_{A \in C(A_1, \ldots, A_n)} \frac{\lambda_p(KA)}{f(A)} = \max_{A} \max_{L} \frac{\lambda_p(KA)}{L(A)}$$

$$= \max_{L} \max_{A} \frac{\lambda_p(KA)}{L(A)} .$$

By Lemma 2,

$$\max_{A} \frac{\lambda_p(KA)}{L(A)} = \frac{\lambda_p(KB)}{L(B)}$$

where B belongs to $H_{\mu_p(2p-\mu_p+1)/2}(A_1,\cdots,A_n)$. To this end

$$\max_{A \in C(A_1,\ldots,A_n)} \frac{\lambda_p(KA)}{f(A)} = \max_{L \in \mathscr{L}} \frac{\lambda_p(KB)}{L(B)} = \frac{\lambda_p(KB)}{f(B)}$$

which concludes the proof of the theorem.

Consider now the maximal problem (4.6) with q linear conditions $h_j(A) = c$ for $j = 1,\cdots,q$.

THEOREM 6. *Let $A_1,\cdots,A_n$ and K be $m \times m$ real valued symmetric non-negative definite matrices with K positive definite. Assume that $h_1,\cdots,h_q$ are q linear functions defined on the set $C(A_1,\cdots,A_n)$ such that at least one matrix from this set satisfies the equalities $h_j(A) = c_j$, for $j = 1,\cdots,q$. Let f be a positive concave function defined on $C(A_1,\cdots,A_n)$. Denote by μ_p the degeneracy index of order p of $C(A_1,\cdots,A_n)$. Then*

$$\max_{A \in C(A_1,\ldots,A_n)} \lambda_p(KA)/f(A) = \max_{B \in H_{\mu_p(2p-\mu_p+1)/2+q}(A_1,\ldots,A_n)} \lambda_p(KB)/f(B) \tag{4.9}$$

for $p = 1,\cdots,m$. The maxumim is taken over the matrices satisfying the q conditions $h_j(A) = c_j$, $j = 1,\cdots,q$.

PROOF. Assume first that $f(A) = 1$ on $C(A_1,\cdots,A_n)$. Then the proof of (4.9) is simply a paraphrasing of the proof of Theorem 4 except that now we have to consider, in addition to the $\nu(2p-\nu+1)/2$ equations (4.5), the q equations

$$\sum_{j=1}^{r} \tilde{\alpha}_j h_k(A_{i_j}) = 0, \quad k = 1,\cdots,q \tag{4.10}$$

in view of the q given conditions. The proof for a general positive concave function f on $C(A_1,\cdots,A_n)$ is merely a repetition of the proof of Theorem 5.

Let $\xi = (\xi_1,\cdots,\xi_n)$ and $\eta = (\eta_1,\cdots,\eta_n)$ be two nonnegative vectors such that

$$0 \leqq \xi_i \leqq \eta_i, \; i = 1,\cdots,n, \; \sum_{i=1}^{n} \xi_i \leqq 1 \leqq \sum_{i=1}^{n} \eta_i. \tag{4.11}$$

Let $C(A_1,\cdots,A_n,\xi,\eta)$ be the following subset of $C(A_1,\cdots,A_n)$:

$$C(A_1,\cdots,A_n,\xi,\eta) = \{A \mid A = \sum_{i=1}^{n} \alpha_i A_i, \; a \in P_n, \;\; \xi_i \leqq \alpha_i \leqq \eta_i, \;\; i = 1,\cdots,n\}.$$

Denote $\xi \leqq \alpha \leqq \eta$ if $\xi_i \leqq \alpha_i \leqq \eta_i$ for all i.

THEOREM 7. (Friedland-Karlin). *Let $A_1,\cdots,A_n$ and K be $m\times m$ real valued symmetric matrices with K positive definite. Assume that ξ and η satisfy conditions* (4.11). *Denote by μ_p the degeneracy index of order p of $C(A_1,\cdots,A_n)$. Let $h_1,\cdots,h_q$ be q linear functions defined on $C(A_1,\cdots,A_n)$. Consider $\max\lambda_p(KA)$ on the set $C(A_1,\cdots,A_n,\xi,\eta)$ subject to q conditions $h_j(A)=c_j$, $j=1,\cdots,q$. Then this maximum is achieved for some $A^*=\sum_{i=1}^n\alpha_i^*A_i$, $\alpha^*\in P_n$, $\xi\leqq\alpha^*\leqq\eta$ and at most $\mu_p(2p-\mu_p+1)/2+q$ of the coefficients of α_i^* are unequal to one of the bounds ξ_i or η_i.*

PROOF. Assume first that $q=0$. Let $\mathscr{E}_p$ be the set of all extremal matrices in $C(A_1,\cdots,A_n,\xi,\eta)$. Let $\alpha=(\alpha_1,\cdots,\alpha_n)$ be a probability vector and denote by α^+ an increasing rearrangement of a vector $a=(a_1,\cdots,a_n)$ where $a_i=\min(\alpha_i-\xi_i,\eta_i-\alpha_i)$.

Define a complete ordering in P_n $\alpha\prec\beta$ by (3.3). Now repeat the proof of Theorem 4. In the case of q linear conditions we add to the equations (4.5) the q equations (4.10) as was done in the proof of Theorem 6.

THEOREM 8 (Friedland-Karlin). *Let $A_1,\cdots,A_n$ and K be $m\times m$ real valued symmetric matrices with K positive definite. Consider*

$$\max_{A\in C(A_1,\ldots,A_n)} f(\lambda_1(KA),\cdots,\lambda_p(KA)) \tag{4.12}$$

where $f(y_1,\cdots,y_p)$ is an increasing function of each of its arguments y_i.

i) *The maximum is achieved for some $A=\sum_{i=1}^n\alpha_iA_i$, $\alpha\in P_n$, involving at most $p(p+1)/2$ of the coefficients of α_i different from zero.*

ii) *If the α_i are subject to the further constraints $\xi_i\leqq\alpha_i\leqq\eta_i$ and q additional linear conditions $\sum_{i=1}^n\alpha_ib_{ij}=c_j$, $j=1,\cdots,q$, then there exists A which maximizes* (4.12) *and has at most $p(p+1)/2+q$ of the coefficients α_i unequal to one of the bounds ξ_i or η_i.*

PROOF. The proof of (i) is similar to the proof of Theorem 1. Let A^* be an extremal matrix of (4.12) $(K=I)$. Let $x_1,\cdots,x_p$ be the p first orthonormal eigenvectors corresponding to $\lambda_1(A^*),\cdots,\lambda_p(A^*)$. Choose A_ε as in the proof of Theorem 1. Now $(A_\varepsilon x,x)=(A^*x,x)$ if a belongs to the subspace $S_j=\mathrm{sp}\{x_1,\cdots,x_j\}$ for $j\leqq p-1$. Thus by the convoy principle

$$\lambda_j(A_\varepsilon)\geqq\min_{x\in S_j}\frac{(A_\varepsilon x,x)}{(x,x)}=\lambda_j(A^*),\quad j=1,\cdots,p-1.$$

If we choose the sign of ε to satisfy (2.8) then we have also the inequality $\lambda_p(A_\varepsilon^*)\geqq\lambda_p(A^*)$. The function $f(y_1,\cdots,y_p)$ by assumption is an increasing function. Furthermore we may assume that $f(y_1,\cdots,y_p)$ is a differentiable function such that $\partial f/\partial y_i > 0$, $i = 1,\cdots,p$. This is no restriction since f may be approximated by such a function. Finally we obtain that

$$f(\lambda_1(A),\cdots,\lambda_p(A_\varepsilon)) \geqq f_1(\lambda_1(A^*),\cdots,\lambda_p(A^*)).$$

Since A^* is an extremal matrix we get a contradiction as in the proof of Theorem 1. The proof of (ii) is simply a modification of the proof of (i) in the same way as in the proof of Theorem 7.

We conclude this section with an analogue result to the equality (4.2) in the case that $A_1,\cdots,A_n$ and K are complex valued hermitian matrices. Noting that the entries of an $m \times m$ complex valued hermitian matrix form m^2 real independent variables we obtain:

THEOREM 9. *Let $A_1,\cdots,A_n$ and K be $m \times m$ complex valued hermitian matrices with K positive definite. Denote by μ_p the degeneracy index of order p of the set $C(A_1,\cdots,A_n)$ with respect to K. Then*

$$\max_{A \in C(A_1,\ldots,A_n)} \lambda_p(KA) = \max_{B \in H\mu_p(2p-\mu_p)(A_1,\ldots,A_n)} \lambda_p(KB) \tag{4.13}$$

for $p = 1,\cdots,m$.

For $\mu_p = p$ the equality (4.13) is best possible. This follows by a proper modification of the example given in Section 2.

5. Integral equations

Let $\mathscr{H}$ be a real Hilbert space. Suppose that A and K are bounded linear symmetric operators defined on $\mathscr{H}$. Assume furthermore that K is positive definite and KA is compact. This requirement is certainly guaranteed if K is compact. Thus the spectrum of KA is at most denumerable with $\lambda = 0$ being the only possible point of accumulation. Denote by $\lambda_1(KA) \geqq \lambda_2(KA) \geqq \cdots$ the positive eigenvalues of KA. If the number of positive eigenvalues is finite, i.e. $\lambda_1(KA) \geqq \lambda_2(KA) \geqq \cdots \geqq \lambda_m(KA) > 0$ then let $\lambda_{m+1}(KA) = \lambda_{m+2}(KA) = \cdots = 0$. Now using the convoy principle and the identities $\lambda_p(KA) = \lambda_p(K^{\frac{1}{2}}AK^{\frac{1}{2}})$, $p=1,2,\cdots$ we realize that all of our results are valid in case that $A_1,\cdots,A_n$ are bounded linear symmetric operators and K is compact and positive definite. As an important application we consider an integral transformation

$$(K\phi)(\xi) = \int_I K(\xi,\eta)\phi(\eta)\sigma(d\eta) \tag{5.1}$$

where I is a compact subset of some Euclidean space and $\sigma(d\eta)$ is a sigma-finite measure on I. The relevant Hilbert space is $L_2(d\sigma)$. Assume that $K(\xi,\eta)$ is symmetric and positive definite. The compactness of the associated operators is guaranteed by the integrability condition

$$\int_I \int_I |K(\xi,\eta)|^2 \sigma(d\xi)\sigma(d\eta) < \infty. \tag{5.2}$$

A bounded linear symmetric operator A to be considered is of the form

$$A\phi(\xi) = a(\xi)\phi(\xi) \tag{5.3}$$

i.e., multiplication of $\phi(\xi)$ by a given nonnegative function $a(\xi)$ belonging to $L^2(d\sigma)$

$$\int_I a^2(\xi)\sigma(d\xi) < \infty. \tag{5.4}$$

Note that A is nonnegative definite. Let $\mathscr{A}$ be a bounded closed set of nonnegative functions $a(\xi)$ in $L_2(d\sigma)$. Assume further that this set is normalized by the condition

$$\int_I a(\xi)u(\xi)\sigma(d\xi) = 1 \tag{5.5}$$

where $u(\xi)$ is a nonnegative continuous function on I. It is well known that $\mathscr{A}$ is a compact set in a weak topology. Let co($\mathscr{A}$) denote the convex closure of $\mathscr{A}$ in a weak topology so co ($\mathscr{A}$) consists of all limits of functions of the form

$$b(\xi) = \sum_{i=1}^{n} \alpha_i a_i(\xi), \ \alpha_i \geqq 0, \ \sum_{i=1}^{n} \alpha_i = 1 \text{ for } a_i(\xi) \in \mathscr{A}.$$

Denote by

$$\lambda_1(a) \geqq \lambda_2(a) \geqq \cdots \geqq 0$$

the nonnegative eigenvalues of the integral equation

$$\int_I K(\xi,\eta)a(\eta)\phi(\eta)\sigma(d\eta) = \lambda\phi(\xi). \tag{5.6}$$

A continuous kernel $K(\xi,\eta)$ is said to be oscillatory if for every collection

$$\xi_1 < \xi_2 < \cdots < \xi_m, \ \eta_1 < \eta_2 < \cdots < \eta_m, \ \xi_i \in I, \ \eta_j \in I$$

the matrix $(K(\xi_i,\eta_j))_1^m$ is oscillatory for $n = 1,\cdots$. It is worth emphasizing that an oscillatory kernel need not be symmetric. An important case is the class of kernels $(-1)^q G(\xi,\eta)$, where G is the Green's function of the differential operator L defined by

$$Ly = \frac{d}{dx}\frac{1}{w_n(x)}\frac{d}{dx}\frac{1}{w_{n-1}(x)}\cdots\frac{d}{dx}\frac{1}{w_1(x)}y(x)$$

where $w_i(x)$, $i = 1,\cdots,n$ are positive of continuity class C^n on $[0,1]$ coupled with the boundary conditions

$$y(0) = y^{(1)}(0) = \cdots y^{(p-1)}(0) = 0,$$
$$y(1) = y^{(1)}(1) = \cdots y^{(q-1)}(1) = 0, \quad p+q=n, \ p,q, \geqq 1,$$

which are nonsingular with respect to L (i.e., only the trivial function satisfies $Ly = 0$ plus the boundary conditions). The demonstration that such kernels are oscillatory including several refinements and extensions involving more general boundary forms is elaborated in [5]. If $K(\xi,\eta)$ is an oscillating kernel and $a(\xi)$ nonnegative nonvanishing functions then

$$\lambda_1(a) > \lambda_2(a) > \cdots > 0$$

(see [3, p. 208]). In that case the set $\mathrm{co}(\mathscr{A})$ is totally nondegenerate. Consider the following maximal problem defined on $\mathrm{co}(\mathscr{A})$:

$$\max \frac{\lambda_p(a)}{[\int_I (a(\xi))^s v(\xi)\sigma(d\xi)]^{1/s}} \tag{5.7}$$

where v is a positive continuous function on I and s is a fixed number $0 < s \leqq$ It is well known that

$$f(a) = \left[\int_I (a(\xi))^s v(\xi)\sigma(d\xi)\right]^{1/s}$$

is a positive concave functional on $\mathrm{co}(\mathscr{A})$. We may impose q linear conditions

$$\int_I a(\xi)w_j(\xi)\sigma(d\xi) = c_j, \quad j = 1,\cdots,q, \tag{5.8}$$

where $w_1(\xi),\cdots,w_q(\xi)$ are continuous functions on I. Theorem 6 can be stated as follows:

THEOREM 10. *Let $\mathscr{A}$ be a bounded and closed set of nonnegative functions in $L_2(d\sigma)$ normalized by the condition* (5.5). *Assume that $K(\xi,\eta)$ is a positive kernel defined on $I \times I$ satisfying the integrability condition* (5.2). *Then the maximum* (5.7) *on $\mathrm{co}(\mathscr{A})$ combined with q linear conditions* (5.8) *is achieved for some function $b(\xi)$ of the form*

(5.9) $$b(\xi) = \sum_{i=1}^{p(p+1)/2+q} \alpha_i a_i(\xi), \ \alpha_i \geqq 0, \ \sum \alpha_i = 1$$

and $a_i(\xi) \in \mathscr{A}$ *for* $i = 1, \cdots, p(p+1)/2 + q$. *Moreover if* $K(\xi, \eta)$ *is a continuous oscillating kernel then at most* $p + q$ *components of the function* $b(\xi)$ (5.9) *do not vanish.*

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